3.1 Quadratic Functions and Models

A **quadratic function** in general form is

\[ f(x) = ax^2 + bx + c \]

where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \).

The **quadratic term** is \( ax^2 \), the **linear term** is \( bx \), and the **constant term** is \( c \). Quadratic functions have degree two. The graph of a quadratic function is called a **parabola**. If \( a < 0 \), then the parabola **opens downward**, like function \( g \). If \( a > 0 \), then the parabola **opens upward** like function \( f \).

If the parabola opens upward, then the **vertex** is the point whose \( y \)-value is the minimum value of \( f \). If the parabola opens downward, then the **vertex** is the point whose \( y \)-value is the maximum value of \( f \). The vertical line that goes through the vertex is called the **axis of symmetry** of the parabola.

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**Example 6**  
**Reading Parabolas**

3. Find \( f(6) \)  

4. Find \( x \) when \( f(x) = 4 \)

5. Find \( g(-2) \)  

6. Find \( x \) when \( g(x) = -8 \)
Polynomial Function – General Form
Let \( a_0, a_1, a_2, a_3, \ldots, a_{n-1}, a_n \) be real numbers and \( n \) be a nonnegative integer and \( a_n \neq 0 \). A polynomial function written in general form with degree \( n \) is a function in the form
\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0
\]

Specific Polynomial Functions
1. A constant function is in the form
\[
f(x) = a_0,
\]
e.g. \( f(x) = -2 \)
has degree zero, and graphs as a horizontal line.

2. A linear function is in the form
\[
f(x) = a_1 x + a_0,
\]
e.g. \( f(x) = 3x - 4 \)
has degree one, and graphs as a straight line with slope \( a_1 \) and \( y \)-intercept \((0, a_0)\).

3. A quadratic function is in the form
\[
f(x) = a_2 x^2 + a_1 x + a_0
\]
e.g. \( f(x) = -3x^2 - 2x + 4 \)
where \( a_2, a_1, \) and \( a_0 \) are real numbers and \( a_2 \neq 0 \). Quadratic functions have degree two and graph as a parabola.

Example 1   Effect of \( a_n \) on the Shape of a Parabola
The graphs of \( f(x) = \frac{1}{3} x^2 \), \( g(x) = x^2 \) and \( h(x) = 2x^2 \) are shown. Notice the only difference is the quadratic coefficient (1/3, 1, and 2) changed. As the quadratic coefficient \(|a|\) gets smaller the graph of the parabola does what?

- Vertically Shrinks (Widens)
- Vertically Stretches (Narrows)
**Example 2**
The graphs of $f(x) = x^2$ is shown.

a. Describe the transformation(s) of $f$ to get $g(x) = (x + 2)^2 - 3$.

b. Sketch the graph of $g(x) = (x + 2)^2 - 3$.

**Example 3**
The graphs of $f(x) = x^2$ is shown.

a. Describe the transformation(s) of $f$ to get $g(x) = -x^2 + 1$.

b. Sketch the graph of $g(x) = -x^2 + 1$.
Quadratic Function in Standard and General Form

A **quadratic function** is a function that can be written in the form

**Standard Form** \( f(x) = a(x - h)^2 + k \)

**General Form** \( f(x) = ax^2 + bx + c \)

where \( a \) is called the **leading coefficient** and

1. The vertex of the parabola is located at \((h, k)\).
2. The equation of the axis of symmetry is \( x = h = -\frac{b}{2a} \).
3. If \( a \) is positive, then the parabola opens upward.
4. If \( a \) is negative, then the parabola opens downward.

**Example 1**  **Find the Vertex Form of a Parabola**

Find the equation for function \( f \) (shown) in vertex form:

**Step 1** Identify the vertex of \( f \). Write the values for \( h \) and \( k \) in

\[ f(x) = a(x - h)^2 + k \]

**Step 2** Identify any other point on \( f \). Use that point to determine \( a \) in

\[ f(x) = a(x - h)^2 + k \].

**Example 2**  **Find the Vertex Form of a Parabola**

Find the equation for function \( g \) (shown above) in vertex form.
Example 4
Let \( f(x) = 2x^2 + 8x + 7 \).

a. Write \( f \) in standard form.

b. Identify the vertex.

c. Identify the axis of symmetry.

d. Find the \( x \)-intercepts to 2 decimal places:

e. Find the \( y \)-intercept and the symmetric point to the \( y \)-intercept.

f. Sketch the graph of \( f \).
Example 5
Let \( f(x) = -x^2 + 6x - 8 \).

a. Write \( f \) in standard form.

b. Identify the vertex.

c. Identify the axis of symmetry.

d. Find the \( x \)-intercepts to 2 decimal places:

e. Find the \( y \)-intercept and the symmetric point to the \( y \)-intercept.

f. Sketch the graph of \( f \).
Example 6
Find the equation of the parabola in standard form whose vertex is (1, 2) and passes through the point (3, -6).

Vertex of a Parabola
Let \( f(x) = ax^2 + bx + c \).

1. The vertex in the graph of \( f \) is \( \left( -\frac{b}{2a}, f\left(-\frac{b}{2a}\right) \right) \).

2. If \( a > 0 \), \( f \) has a minimum value of \( f\left(-\frac{b}{2a}\right) \) at \( x = -\frac{b}{2a} \).

3. If \( a < 0 \), \( f \) has a maximum value of \( f\left(-\frac{b}{2a}\right) \) at \( x = -\frac{b}{2a} \).

Example 7  Maximum Height of a Baseball
A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The pay of the baseball is given by the function
\[ h(x) = -0.0032x^2 + x + 3, \]
where \( h(x) \) is the height of the baseball (in feet) and \( x \) is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?
Example 8  Find the Minimum Cost
A small soft-drink manufacturer has daily production cost of 
\[ C = 70,000 - 120x + 0.075x^2, \]
where \( C \) is the total cost (in dollars) of producing \( x \) units. How many units should be produced each 
day to yield a minimum cost? What is that minimum cost?

Example 7  Grants
The number of grants \( g \) awarded from the National Endowment for 
Humanities fund from 1999 to 2003 can be approximated by the 
model 
\[ g(t) = -99.140t^2 + 2.201t - 10.896, \quad 9 \leq t \leq 13 \]
where \( t \) is the number of years since 1990. Determine the year in 
which the number of grants awarded was greatest.
3.2 Polynomial Functions of Higher Degree

The graphs of all polynomials are (1) continuous and (2) have only smooth rounded curves. The simplest polynomial function is the power function given as

\[ f(x) = x^n, \quad \text{where } n \text{ is a positive integer.} \]

Power functions are either odd or even.

**Even Power Functions**

The graphs of the functions \( f(x) = x^2 \), \( g(x) = x^4 \) and \( h(x) = x^6 \) are shown.

a. On the interval \([-1, 1]\), as the exponent \( n \) increases the graph gets __________.

b. Outside the interval \([-1, 1]\), as the exponent \( n \) increases the graph gets __________.

c. The graph of an even power function of the form \( f(x) = x^n \) [rises, falls] on the far right and [rises, falls] on the far left.

d. The graph of an even power function of the form \( f(x) = -x^n \) is a reflection \( y = x^n \) across the [x-axis, y-axis], and its graph [rises, falls] on the far right and [rises, falls] on the far left.
Odd Power Functions
The graphs of the functions \( f(x) = x^3 \), \( g(x) = x^5 \) and \( h(x) = x^7 \) are shown.

a. On the interval \([-1, 1]\), as the exponent \( n \) increases the graph gets
   ________________.

b. Outside the interval \([-1, 1]\), as the exponent \( n \) increases the graph gets
   ________________.

c. The graph of an odd power function of the form \( f(x) = x^n \) [rises, falls] on the far right and [rises, falls] on the far left.

d. The graph of an odd power function of the form \( f(x) = -x^n \) is a reflection \( y = x^n \) across the [x-axis, y-axis], and its graph [rises, falls] on the far right and [rises, falls] on the far left.
The Leading Coefficient Test
As \( x \to \infty \) or \( x \to -\infty \), the graph of the polynomial function
\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0
\]
eventually rises or falls on the far right and far left in the following manner.

1. When \( n \) is odd the graph falls on one side and rises on the other.

   ![Graphs for odd n](image1.png)

   If the leading coefficient is positive \( (a_n > 0) \), the graph falls on the left and rises on the right. 
   If the leading coefficient is negative \( (a_n < 0) \), the graph falls on the right and rises on the left.

1. When \( n \) is even the graphs rises on both sides or falls on both sides.

   ![Graphs for even n](image2.png)

   If the leading coefficient is positive \( (a_n > 0) \), the graph rises on both sides. 
   If the leading coefficient is positive \( (a_n > 0) \), the graph falls on both sides.
Number of Zeros and Turning Points on Polynomials

For a polynomial function $f$ with degree $n$, we can say
1. The function $f$ has precisely $n$ zeros (counting multiplicities)
2. The function $f$ has, at most, $n - 1$ turning points (aka relative extrema).

Example 2

(1) Describe the right and left hand behavior of the graph of each function. (2) Describe the possible number of turning points and the number of zeros for each function. (3) Classify each function as even, odd or neither.

a. $f(x) = -x^3 + 4x = -x(x+2)(x-2)$

The graph of $f$ ________ on the left and ________ on the right.
The graph of $f$ has at most ________ turning points.
$f$ is an ________ function and is symmetric with respect to ________.

b. $f(x) = x^4 - 5x^2 + 4 = (x+2)(x-2)(x+1)(x-1)$

The graph of $f$ ________ on the left and ________ on the right.
The graph of $f$ has at most ________ turning points.
$f$ is an ________ function and is symmetric with respect to ________.

c. $f(x) = x^5 - x = x(x+1)(x-1)(x^2+1)$

The graph of $f$ ________ on the left and ________ on the right.
The graph of $f$ has at most ________ turning points.
$f$ is an ________ function and is symmetric with respect to ________.
Real Zeros of Polynomial Functions

For a polynomial function \( f \) and real number \( a \), the following statements are equivalent.
1. \( x = a \) is a zero of \( f \).
2. \( x = a \) is a solution of the polynomial \( f(x) = 0 \).
3. \( (x - a) \) is a factor of the polynomial \( f(x) \).
4. \( (a, 0) \) is an \( x \)-intercept of the graph of \( f \).

Example 4

a. Describe the graph of \( f(x) = -x^3 + 4x = -x(x+2)(x-2) \).

| Zeros of \( f \): | ______________________ |
| \( f(x) \) in factored form: | ______________________ |
| \( x \)-intercepts of \( f \): | ______________________ |
| \( y \)-intercept of \( f \): | ______________________ |
| Right-hand behavior: | ______________________ |
| Left-hand behavior: | ______________________ |
| Number of turning points: | ______________________ |
| Relative Maximum(s): | ______________________ |
| Relative Minimum(s): | ______________________ |
| Is \( f \) even, odd or neither? | ______________________ |

b. Sketch the graph of \( f(x) = -x^3 + 4x \)
Example 5

a. Describe the graph of \( f(x) = x^4 - 5x^2 + 4 \)

- Zeros of \( f \): ___________________
- \( f(x) \) in factored form: ___________________
- \( x \)-intercepts of \( f \): ___________________
- \( y \)-intercept of \( f \): ___________________
- Right-hand behavior: ___________________
- Left-hand behavior: ___________________
- Number of turning points: ___________________
- Relative Maximum(s): ___________________
- Relative Minimum(s): ___________________
- Is \( f \) even, odd or neither? ___________________

b. Sketch the graph of \( f \).
Example 4

a. Describe the graph of \( f(x) = x^5 - x \)

Real zeros of \( f \): ___________________

\( f(x) \) in factored form: ___________________

\( x \)-intercepts of \( f \): ___________________

\( y \)-intercept of \( f \): ___________________

Right-hand behavior: ___________________

Left-hand behavior: ___________________

Number of turning points: ___________________

Is \( f \) even, odd or neither? ___________________

Relative Minimum: ___________________

Relative Maximum: ___________________

b. Sketch the graph of \( f \).
Example 5
Describe the graph of $f(x) = -2x^4 + 2x^2$

Real zeros of $f$: ___________________

$f(x)$ in factored form: ___________________

$x$-intercepts of $f$: ___________________

$y$-intercept of $f$: ___________________

Right-hand behavior: ___________________

Left-hand behavior: ___________________

Number of turning points: ___________________

Is $f$ even, odd or neither? ___________________

Relative Minimum: ___________________

Relative Maximum: ___________________

b. Sketch the graph of $f$. 

[Graph of $f(x) = -2x^4 + 2x^2$]
Repeated Real Zeros and Multiplicity of Zeros
For \( k > 1 \), a factor \((x - a)^k\) yields a **repeated real zero** of **multiplicity** \( k \).

1. If \( k \) is odd, the graph crosses the \( x \)-axis at \( x = a \).
2. If \( k \) is even, the graph touches the \( x \)-axis (but does not cross) at \( x = a \).

**Example 6**
Classify the zeros of \( f(x) = -2x^4 + 2x^2 = -2x^2(x - 1)(x + 1) \):

<table>
<thead>
<tr>
<th>Zero</th>
<th>Multiplicity</th>
<th>Touches or Crosses the ( x )-axis</th>
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</table>

**Example 7**
Classify the zeros of \( f(x) = x^5 - x = x(x^2 + 1)(x + 1)(x - 1) \):

<table>
<thead>
<tr>
<th>Real Zero</th>
<th>Multiplicity</th>
<th>Touches or Crosses the ( x )-axis</th>
</tr>
</thead>
</table>
Example 8

a. Analyze the graph of $f(x) = 3x^4 - 4x^3$.

Write $f(x)$ in factored form: ___________________

Real zeros of $f$: ___________________

Imaginary zeros of $f$: ___________________

$x$-intercepts of $f$: ___________________

$y$-intercept of $f$: ___________________

Right-hand behavior: ___________________

Left-hand behavior: ___________________

Number of turning points: ___________________

Is $f$ even, odd or neither? ___________________

Relative Minima: ___________________

Relative Maxima: ___________________

b. Sketch the graph of $f(x) = 3x^4 - 4x^3$. 

![Graph of $f(x) = 3x^4 - 4x^3$]
Example 9

a. Analyze the graph of \( f(x) = -2x^3 + 6x^2 - \frac{9}{2}x \).

Write \( f(x) \) in factored form: __________________

Real zeros of \( f \): __________________

Imaginary zeros of \( f \): __________________

\( x \)-intercepts of \( f \): __________________

\( y \)-intercept of \( f \): __________________

Right-hand behavior: ________________

Left-hand behavior: ________________

Number of turning points: ________________

Is \( f \) even, odd or neither? ________________

Relative Minima: __________________

Relative Maxima: __________________

b. Sketch the graph of \( f(x) = -2x^3 + 6x^2 - \frac{9}{2}x \).
**Intermediate Value Theorem**

Let $a$ and $b$ be real numbers such that $a < b$. If $f$ is a polynomial function and $f(a) \neq f(b)$, then, in the interval $[a, b], f$ takes on every value between $f(a)$ and $f(b)$.

**Corollary to the Intermediate Value Theorem**

Let $a$ and $b$ be real numbers such that $a < b$ and $f$ be a polynomial function. If $f(a)$ and $f(b)$ have different signs, then, in the interval $[a, b], f$ has a zero.

**Example 10**

The graph of $f(x) = x^3 + x^2 + 1$ is shown. Use the Intermediate Value Theorem to show that $f$ has a zero on $[-2, -1]$. 
3.3 Real Zeros of Polynomial Functions

Example 1
Let \( f(x) = 6x^3 - 19x^2 + 16x - 4 \)

a. Find \( f(2) \). What does this mean in terms of writing \( f \) in factored form.

b. Divide \( f(x) \) by \( x - 2 \).

c. Write \( f(x) = 6x^3 - 19x^2 + 16x - 4 \) in factored form.
The Division Algorithm
Let \( f(x) \) and \( d(x) \neq 0 \) be polynomials where the degree of \( d(x) \) is less than the degree of \( f(x) \). There exist unique polynomials \( q(x) \) and \( r(x) \) so that

\[
\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \quad \text{or} \quad f(x) = q(x) \cdot d(x) + r(x),
\]

where \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( d(x) \). Moreover, if \( r(x) = 0 \), then \( d(x) \) divides evenly into \( f(x) - \) that is \( d(x) \) and \( q(x) \) are factors of \( f(x) \).

Example 2
Let \( f(x) = 2x^4 + 4x^3 - 5x^2 + 3x - 2 \)

a. Divide \( f(x) \) by \( x^2 + 2x - 3 \).

b. Write \( f \) in the form \( f(x) = q(x) \cdot d(x) + r(x) \).
Example 3
Let \( f(x) = x^3 - 1 \)

a. Divide \( f(x) \) by \( x - 1 \).
b. Write \( f \) in the form \( f(x) = q(x) \cdot d(x) + r(x) \).

Synthetic Division when Dividing by \((x - k)\)

Example 4
Let \( f(x) = x^3 - 1 \)

a. Use synthetic division to divide \( f(x) \) by \( x - 1 \).
b. Write \( f \) in the form \( f(x) = q(x) \cdot d(x) + r(x) \).
c. Find \( f(1) = f(k) \)
Example 5
Let \( f(x) = x^4 - 10x^2 - 2x + 4 \).

a. Use synthetic division to divide \( f(x) \) by \( x + 3 \).

b. Write \( f \) in the form \( f(x) = q(x) \cdot d(x) + r(x) \).

c. Find \( f(-3) \)

The Remainder Theorem
If \( f(x) \) is divided by \( x - k \), then the remainder \( r \) is a constant and \( r = f(k) \). That is, \( f(x) = (x - k) \cdot q(x) + r \).

Example 6
Let \( f(x) = 3x^3 - 8x^2 + 5x - 7 \).

a. Use the remainder theorem to find \( f(-2) \).

b. Use the remainder theorem to find \( f(3) \).
The Factor Theorem
A polynomial \( f(x) \) has a factor \( x - k \) if and only if \( f(k) = 0 \).

Example 7
Let \( f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18 \).

Show that \( (x - 2) \) and \( (x + 3) \) are factors of \( f \). Write \( f \) as a product of linear factors.

Rational Zero Test
If the polynomial \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \) has integer coefficients, then every rational zero of \( f \) has the form

\[
\text{Rational zero} = \frac{p}{q} = \frac{\text{factor of the constant term } a_0}{\text{factor of the leading coefficient } a_n},
\]

where \( p \) and \( q \) have no common factors other than 1, and

\[
p = \text{a factor of the constant term } a_0 \\
q = \text{factor of the leading coefficient } a_n
\]

Example 8
Use the Rational Zero Test to list all the possible rational zeros of \( f(x) = x^3 + x + 1 \). Then find the rational zeros of \( f \).

Possible Rational Zeros: ________________________

Actual rational zeros of \( f \): ________________________

Write \( f \) in factored form: ________________________
Example 9
Use the Rational Zero Test to list all the possible rational zeros of 
\( f(x) = 2x^3 + 3x^2 - 8x + 3 \). Then find all the zeros of \( f \) and write \( f \) as a product of linear factors.

Possible Rational Zeros: ________________

Actual rational zeros of \( f \): ________________

Write \( f \) in factored form: ________________

Example 10
Use the Rational Zero Test to list all the possible rational zeros of 
\( f(x) = x^4 - x^3 + x^2 - 3x - 6 \). Then find the following:

Possible Rational Zeros: ________________

Actual rational zeros of \( f \): ________________

Write \( f \) in factored form: ________________
Example 9
A rectangular package with a square base can be delivered by a delivery service if maximum combined length and girth (perimeter of the cross section) is 120 inches.

a. Find the equation for volume in terms of \( x \).
b. Find the dimensions of the package that will yield a maximum volume.
c. Find the value of \( x \) so that \( V = 13,500 \). Explain the solution(s) in this application.
3.4 The Fundamental Theorem of Algebra

If \( f(x) \) is a polynomial of degree \( n > 0 \), then \( f \) has at least one zero in the complex number system.

**Linear Factorization Theorem**

If \( f(x) \) is a polynomial of degree \( n > 0 \), then \( f \) has precisely \( n \) linear factors and can be written in the form

\[
f(x) = a_n (x - c_1)(x - c_2) \cdots (x - c_n),
\]

where \( c_1, c_2, \ldots, c_n \) are zeros of \( f \).

**Example 1**

Identify the leading coefficient \( a_n \) and zeros \( c_1, c_2, \ldots, c_n \) of polynomial \( f \). Then write \( f \) in as a product of linear factors.

a. \( f(x) = x - 2 \)

b. \( f(x) = x^3 + 4x \)

c. \( f(x) = x^4 - 1 \)
Example 2
Write \( f(x) = x^5 + x^3 + 2x^2 - 12x + 8 \) as a product of linear factors and list all the zeros of \( f \).

Complex Zeros Occur in Conjugate Pairs
Let \( f \) be a polynomial with real coefficients. If \( a + bi \) is a zero of \( f \), then the conjugate \( a - bi \) is also a zero of \( f \).

Example 3
a. Find a fourth-degree polynomial with real coefficients that has the zeros -1 (with multiplicity 2) and 3i. Recall \( f(x) = a_n(x - c_1)(x - c_2)(x - c_3)(x - c_4) \).

b. Write \( f \) as a product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.
Factors of a Polynomial with Real Coefficients
Every polynomial of degree \( n > 0 \) with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros. A quadratic factor with no real zeros is said to be **irreducible over the real numbers**.

Example 4
Given \( f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60 \) has zeros -2, 3, 1 + 3i, 1 – 3i.

a. Write \( f \) as a product of linear factors.

b. Write \( f \) as a product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.
Example 7
Let \( f(x) = x^5 + x^3 + 2x^2 - 12x + 8 \).

1. Find the zeros of \( f \).

2. Write \( f \) as a product of linear factors.

3. Find the \( x \)- and \( y \)-intercepts in the graph of the function.

4. Estimate the relative maxima and relative minima in the graph of the \( f \).

5. Find any other points that may help in sketching the graph. Choose an appropriate viewing rectangle. Then sketch the graph of the \( f \). Label and scale the axes.
Example 8
Let \( f(x) = 3x^3 - 5x^2 + 6x - 4 \).

1. Find the zeros of \( f \).

2. Write \( f \) as a product of linear factors.

3. Find the \( x \)- and \( y \)-intercepts in the graph of the function.

4. Estimate the relative maxima and relative minima in the graph of the \( f \).

5. Find any other points that may help in sketching the graph. Choose an appropriate viewing rectangle. Then sketch the graph of the \( f \). Label and scale the axes.
3.5 Rational Functions and Asymptotes

Rational Function
If $N(x)$ and $D(x)$ are polynomials, then a rational function $f$ is a function that can be written in the form

$$f(x) = \frac{N(x)}{D(x)}, \text{ where } D(x) \neq 0.$$ 

Some examples of rational functions are

$$f(x) = \frac{x^3 - 3x}{x - 8}, \quad g(x) = \frac{-2x^2 + 17}{5x - 1}, \quad h(x) = -\frac{4}{5x^3}.$$ 

Domain of a Rational Function
The domain of the rational function $f(x) = \frac{N(x)}{D(x)}$ is the set of all real numbers except where $D(x) = 0$.

Example 1  Find the domain of

a. $f(x) = \frac{x^3 - 3x}{x - 8}$  
   b. $g(x) = \frac{-2x^2 + 17}{5x - 1}$  
   c. $h(x) = -\frac{4}{5x^3}$
Vertical Asymptote
A **vertical asymptote** is a vertical line that a graph of a function gets arbitrarily close to, but never touches nor crosses. For example, the graph of \( f \) in figure 1 has a vertical asymptote at \( x = 5 \).

Vertical asymptotes only appear at values of \( x \) that make the denominator zero. Since \( f(5) \) is undefined, choose \( x \)-values that “get close to 5.”

<table>
<thead>
<tr>
<th>( x \rightarrow 5^- ) from the left</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>4.5</th>
<th>4.9</th>
<th>4.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \rightarrow -\infty )</td>
<td>-0.6</td>
<td>-1</td>
<td>-3</td>
<td>-6</td>
<td>-30</td>
<td>-300</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x \rightarrow 5^+ ) from the right</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5.5</th>
<th>5.1</th>
<th>5.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \rightarrow \infty )</td>
<td>1</td>
<td>1.5</td>
<td>3</td>
<td>6</td>
<td>30</td>
<td>300</td>
</tr>
</tbody>
</table>

**Figure 1** Graph of \( f(x) = \frac{3}{x - 5} \)

Horizontal Asymptote
A **horizontal asymptote** is a horizontal line that the graph of a function gets arbitrarily close to as “\( x \) approaches infinity” (\( x \rightarrow \pm \infty \))

<table>
<thead>
<tr>
<th>( x \rightarrow -\infty )</th>
<th>-2</th>
<th>-10</th>
<th>-200</th>
<th>-500</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \rightarrow 0 )</td>
<td>-0.42</td>
<td>-0.200</td>
<td>-0.146</td>
<td>0.006</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x \rightarrow \infty )</th>
<th>10</th>
<th>20</th>
<th>200</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \rightarrow 0 )</td>
<td>0.6</td>
<td>0.2</td>
<td>0.015</td>
<td>0.006</td>
</tr>
</tbody>
</table>
**Definition of Horizontal and Vertical Asymptotes**

1. A line \( x = a \) is a **vertical asymptote** of the graph of \( f \) if 
   \[ f(x) \to \infty \text{ or } f(x) \to -\infty \text{ as } x \to a^+ \text{ or } x \to a^- \]

2. A line \( y = b \) is a **horizontal asymptote** of the graph of \( f \) if 
   \[ f(x) \to b \text{ as } x \to \pm \infty \].

**Example 1**

The graph of \( f(x) = \frac{2x+1}{x+1} \) is shown.

a. Discuss the domain how to find the vertical asymptote(s) of \( f \).

b. Discuss how to find a horizontal asymptote.
Example 2

The graph of $f(x) = \frac{4}{x^2 + 1}$ is shown.

a. Discuss the domain and how to find the vertical asymptote(s) of $f$.

b. Discuss how to find a horizontal asymptote.

Example 3

The graph of $f(x) = \frac{2}{(x-1)^2}$ is shown.

a. Discuss the domain and how to find the vertical asymptote(s) of $f$.

b. Discuss how to find a horizontal asymptote.
Asymptotes of Rational Functions

Let \( f \) be the rational function given by

\[
f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0},
\]

where \( N(x) \) and \( D(x) \) have no common factors, the degree of the numerator \( N(x) \) is \( n \), and the degree of the denominator \( D(x) \) is \( m \).

1. **Vertical Asymptotes** The graph of \( f \) has a vertical asymptote at each zero of \( D(x) \).

2. **Horizontal & Slant Asymptotes** The graph of \( f \) has one or no horizontal asymptote determined by the degrees of \( N(x) \) and \( D(x) \).
   a. If \( n < m \), then \( f \) has the line \( y = 0 \) (the \( x \)-axis) as a horizontal asymptote.
   b. If \( n = m \), then \( f \) has the line \( y = a_n / b_m \) (the ratio of the leading coefficients) as a horizontal asymptote.
   c. If \( n = m + 1 \), then \( f \) has a slant asymptote whose equation is the quotient polynomial in the division algorithm.
   d. If \( n > m + 1 \), then \( f \) has no horizontal or slant asymptote.

\[
a. f(x) = \frac{2x}{4x^2 - 9} \\
b. f(x) = \frac{2x^2}{4x^2 - 9} \\
c. f(x) = \frac{2x^3}{4x^2 - 9}
\]
Example 4  For \( f(x) = \frac{x^2 + 2x - 2}{x^2 - x - 6} \). Find each of the following.

a. The domain of \( f \).

b. The [equation of the] horizontal asymptotes in the graph of \( f \).

c. The [equation of the] vertical asymptotes in the graph of \( f \).

d. The location of any holes in the graph of \( f \).

e. The \( x \)-intercepts in the graph of \( f \).

f. The \( y \)-intercepts in the graph of \( f \).

Example 5  For \( f(x) = \frac{3x^2 + 7x + 2}{-4x^2 + 5} \). Find each of the following.

a. The domain of \( f \).

b. The [equation of the] horizontal asymptotes in the graph of \( f \).

c. The [equation of the] vertical asymptotes in the graph of \( f \).

d. The location of any holes in the graph of \( f \).

e. The \( x \)-intercepts in the graph of \( f \).

f. The \( y \)-intercepts in the graph of \( f \).
Example 6
A Graph with Two Horizontal Asymptotes
Find the horizontal asymptotes in the graph of \( f(x) = \frac{x + 10}{|x| + 2} \).

Example 7
A utility \( C \) (in dollars) of removing \( p\% \) of the smokestack pollutants is given by \( C = \frac{80,000p}{(100 - p)} \) for \( 0 \leq p < 100\% \). The legislature is considering a law that would mandate 90\% of the pollutants be removed (up from the current 85\%). How much additional cost would the utility company incur as a result of the new law?
Example 8  Ultraviolet Radiation
For a person with sensitive skin, the amount of time \( T \) (in hours) he can be exposed to the sun with minimal burning can be modeled by

\[
T = \frac{0.37s + 23.8}{s}, \quad 0 < s < 120
\]

where \( s \) is the Sunsor Scale reading. The Sunsor Scale is based on the level of intensity of the UVB rays.

a. Find the amount of time a person with sensitive skin can be exposed to the sun with minimal burning when \( s = 1, 10, 25, 100 \) and \( 120 \).

<table>
<thead>
<tr>
<th>( s )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td></td>
</tr>
</tbody>
</table>

b. If the model were valid for all \( s > 0 \), what would be the horizontal asymptote of this function, and what would it mean?
3.6 Graphs of Rational Functions

Guidelines to Analyze Graphs of Rational Functions
Let \( f(x) = \frac{N(x)}{D(x)} \), where \( D(x) \neq 0 \), be a rational function.

1. **Simplify \( f \)** Simplify \( f \), if possible - so that \( N(x) \) and \( D(x) \) have no common factors other than 1. Identify any “holes” in the graph of \( f \).

2. **\( y \)-intercept** Find and plot the \( y \)-intercept (if any) by evaluating \( f(0) \).

3. **\( x \)-intercept(s)** Find the zeros of the numerator, by solving \( N(x) = 0 \). Then plot the corresponding \( x \)-intercepts.

4. **Vertical Asymptotes** Find the zeros of the denominator by solving \( D(x) \neq 0 \). Then dash-in the corresponding vertical asymptote(s).

5. **Horizontal/Slant Asymptote** Analyze the degrees of \( N(x) \) and \( D(x) \) to find any horizontal or slant asymptote. Dash-in any horizontal/slant asymptote.

6. **Additional Points** Plot at least one point between and beyond *each* \( x \)-intercept and vertical asymptote.

7. **Sketch** Use smooth curves to complete the graph.

**Example 1**

Sketch the graph of \( f(x) = \frac{3}{x - 2} \).

\( y \)-intercept: __________________

\( x \)-intercept(s): __________________

Vertical asymptote(s): __________________

Horizontal asymptote: __________________

Additional points:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2

Sketch the graph of \( f(x) = \frac{2x - 1}{x} \).

\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
x & -5 & -3 & -1 & 1 & 3 & 5 \\
\hline
f(x) & 2.2 & 2.3 & 3 & 1 & 1.7 & 1.8 \\
\hline
\end{array} \]

y-intercept: 
\[ \quad \]

x-intercept(s): 
\[ \quad \]

Vertical asymptote(s): 
\[ \quad \]

Horizontal asymptote: 
\[ \quad \]

Additional points:

Example 3

Sketch the graph of \( f(x) = \frac{x}{x^2 - x - 2} \).

\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
x & -5 & -3 & 0 & 1 & 3 & 5 \\
\hline
f(x) & -0.2 & -0.3 & 0 & -0.5 & 0.75 & 0.3 \\
\hline
\end{array} \]

y-intercept: 
\[ \quad \]

x-intercept(s): 
\[ \quad \]

Vertical asymptote(s): 
\[ \quad \]

Horizontal asymptote: 
\[ \quad \]

Additional points:
Example 4  A Hole in the Graph

Sketch the graph of \( f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} \).

y-intercept: __________________

x-intercept(s): ________________

Vertical asymptote(s): __________

Horizontal asymptote: __________

Additional points:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-3</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.5</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1.7</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Slant Asymptotes

In a rational function \( f(x) = \frac{N(x)}{D(x)} \) where the degree of the numerator is precisely one more than the degree of the denominator, then \( f \) has a slant asymptote whose equation is the quotient polynomial from the division algorithm. For example,

\[
f(x) = \frac{x^2 - x}{x + 1} = x - 2 + \frac{2}{x + 1}
\]

has a vertical asymptote at \( x = -1 \) and a slant asymptote whose equation is \( y = x - 2 \).
Guidelines to Analyze Graphs of Rational Functions

Let \( f(x) = \frac{N(x)}{D(x)} \), where \( D(x) \neq 0 \), be a rational function.

1. **Simplify \( f \)**  
   Simplify \( f \), if possible - so that \( N(x) \) and \( D(x) \) have no common factors other than 1. Identify any “holes” in the graph of \( f \).

2. **\( y \)-intercept**  
   Find and plot the \( y \)-intercept (if any) by evaluating \( f(0) \).

3. **\( x \)-intercept(s)**  
   Find the zeros of the numerator, by solving \( N(x) = 0 \). Then plot the corresponding \( x \)-intercepts.

4. **Vertical Asymptotes**  
   Find the zeros of the denominator by solving \( D(x) \neq 0 \). Then dash-in the corresponding vertical asymptote(s).

5. **Horizontal/Slant Asymptote**  
   Analyze the degrees of \( N(x) \) and \( D(x) \) to find any horizontal or slant asymptote. Dash-in any horizontal/slant asymptote.

6. **Additional Points**  
   Plot at least one point between and beyond each \( x \)-intercept and vertical asymptote.

7. **Sketch**  
   Use smooth curves to complete the graph.

**Example 5  
Graph with a Slant Asymptote**

Sketch the graph of \( f(x) = \frac{x^2 - x - 2}{x - 1} \).

\( y \)-intercept: ________________________

\( x \)-intercept(s): ________________________

Vertical asymptote(s): ________________________

Horizontal/Slant asymptote: ________________________

Additional points:

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(-3.6)</td>
<td>(-2.5)</td>
<td>(-1.3)</td>
<td>(0)</td>
<td>(2)</td>
<td>undef</td>
<td>(0)</td>
<td>(2)</td>
<td>(3.3)</td>
</tr>
</tbody>
</table>
Example 6  Graph with a Slant Asymptote

Sketch the graph of \( f(x) = \frac{x^3}{2x^2 - 8} \).

<table>
<thead>
<tr>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.67</td>
</tr>
</tbody>
</table>

Example 7  Minimize Paper Used

A rectangular page is designed to contain 48 square inches of print. The margins at the top and bottom of the page are 1 inch each, and the side margins are each 1.5 inches. What dimensions should the page be so that the least amount of paper is used?
3.7 Quadratic Models

Example 1
A study was done to compare the speed $x$ (mph) with the mileage $y$ (mpg) of an automobile. Find the appropriate regression equation (linear or quadratic) that models the data. Round the constants to 4 decimal places.

<table>
<thead>
<tr>
<th>Speed</th>
<th>Milage</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>22.3</td>
</tr>
<tr>
<td>20</td>
<td>25.5</td>
</tr>
<tr>
<td>25</td>
<td>27.5</td>
</tr>
<tr>
<td>30</td>
<td>29.0</td>
</tr>
<tr>
<td>35</td>
<td>28.8</td>
</tr>
<tr>
<td>40</td>
<td>30.0</td>
</tr>
<tr>
<td>45</td>
<td>29.9</td>
</tr>
<tr>
<td>50</td>
<td>30.2</td>
</tr>
<tr>
<td>55</td>
<td>30.4</td>
</tr>
<tr>
<td>60</td>
<td>28.8</td>
</tr>
<tr>
<td>65</td>
<td>27.4</td>
</tr>
<tr>
<td>70</td>
<td>25.3</td>
</tr>
<tr>
<td>75</td>
<td>23.3</td>
</tr>
</tbody>
</table>

Example 2
The table shows the sales $S$ (in millions of dollars) for running shoes from 1998 to 2004. Find the appropriate regression equation (linear or quadratic) that models the data. Round the constants to 4 decimal places.

<table>
<thead>
<tr>
<th>Year</th>
<th>Sales ($\text{millions}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>1469</td>
</tr>
<tr>
<td>1999</td>
<td>1502</td>
</tr>
<tr>
<td>2000</td>
<td>1638</td>
</tr>
<tr>
<td>2001</td>
<td>1670</td>
</tr>
<tr>
<td>2002</td>
<td>1733</td>
</tr>
<tr>
<td>2003</td>
<td>1802</td>
</tr>
<tr>
<td>2004</td>
<td>1838</td>
</tr>
</tbody>
</table>