7.1 Graphs of Quadratic Functions in Vertex Form

Quadratic Function in Vertex Form
A quadratic function in vertex form is a function that can be written in the form
\[ f(x) = a(x - h)^2 + k \]
where \( a \) is called the leading coefficient and
1. The vertex of the parabola is located at \((h, k)\).
2. The equation of the axis of symmetry is \( x = h \).
3. If \( a \) is positive, then the parabola opens upward.
4. If \( a \) is negative, then the parabola opens downward.

Example 1  Find the Vertex Form of a Parabola
Find the equation for function \( f \) (shown) in vertex form:

Step 1  Identify the vertex of \( f \). Write the values for \( h \) and \( k \) in
\[ f(x) = a(x - h)^2 + k \]

Step 2  Identify any other point on \( f \). Use that point to determine \( a \) in
\[ f(x) = a(x - h)^2 + k \]

Example 2  Find the Vertex Form of a Parabola
Find the equation for function \( g \) (shown above) in vertex form.
### Example 2  Vertex Form of a Parabola

Fill-in the table and match each function to its graph.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Function</th>
<th>$h$</th>
<th>$k$</th>
<th>$a$</th>
<th>Vertex</th>
<th>Axis of Symmetry</th>
<th>Opens Upward</th>
<th>Opens Downward</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f(x) = -3(x - 2)^2 + 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f(x) = 2(x + 3)^2 - 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f(x) = -(x - 7)^2 + 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f(x) = (x - 2)^2 + 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f(x) = 4x^2 + 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f(x) = x^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 3   The Effect of $a$ on the Shape of a Parabola

1. The graphs of $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = (1/3)x^2$ are shown.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$h(x) = \frac{1}{3}x^2$</th>
<th>$f(x) = x^2$</th>
<th>$g(x) = 2x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>3</td>
<td>9</td>
<td>18</td>
</tr>
<tr>
<td>-2</td>
<td>1.33</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>-1</td>
<td>0.33</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.33</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1.33</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>9</td>
<td>18</td>
</tr>
</tbody>
</table>

2. Compared to $f$, the graph of $g$ is
   a. narrower
   b. wider

3. As the leading quadratic coefficient $a$ gets larger and larger, what happens to the shape of the graph?
   a. The graph gets wider and wider
   b. The graph gets narrower and narrower

4. Compared to $f$, the graph of $h$ is
   a. narrower
   b. wider

5. As the leading quadratic coefficient $a$ gets closer to zero, what happens to the shape of the graph?
   a. The graph gets wider and wider
   b. The graph gets narrower and narrower
Graphs of Parabolas in the Form $f(x) = ax^2$

1. The graph is a parabola with vertex at $(0, 0)$.
2. As $|a|$ gets larger and larger, the graph of $f$ narrows.
3. As $a$ gets closer to zero, the graph of $f$ widens.
4. If $a$ is positive, then the graph of $f$ opens upward.
5. If $a$ is negative, then the graph of $f$ opens downward.
6. The graphs of $y = ax^2$ and $y = -ax^2$ are reflections of each other across the $x$-axis.

**Example 4** Reflecting a Graph across the $x$-axis

Sketch the graphs of $f(x) = \frac{1}{2}x^2$ and $g(x) = -\frac{1}{2}x^2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = \frac{1}{2}x^2$</th>
<th>$g(x) = -\frac{1}{2}x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Graph of Parabolas]

- The table and graph illustrate how the parabolas are reflected across the $x$-axis:
- The graph of $f(x)$ is a parabola opening upward.
- The graph of $g(x)$ is a parabola opening downward, reflecting $f(x)$ across the $x$-axis.
Example 5  **Shapes & Reflections of Parabolas**
Write reasonable equations for the family of parabolas shown.
### Example 6  Vertical Translations (Up & Down Shifts)

The graph of $f(x) = x^2$ is shown. Complete the table and sketch the graphs of $g(x) = x^2 - 3$ and $h(x) = x^2 + 5$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
<th>$g(x) = x^2 - 3$</th>
<th>$h(x) = x^2 + 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Example 7  Horizontal Translations

The graph of $f(x) = x^2$ is shown. Complete the table and sketch the graph of $g(x) = (x - 5)^2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
<th>$g(x) = (x - 5)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>
Steps to Graph a Quadratic Function

To sketch the graph of \( f(x) = a(x - h)^2 + k \).

1. Sketch the graph of \( y = ax^2 \).
2. Translate the graph from step 1 by \( h \) units horizontally.
3. Translate the graph from step 2 by \( k \) units vertically.

Example 8

1. The graph of \( f(x) = 0.5x^2 \) is shown.
   Sketch the graph of \( g(x) = 0.5(x - 3)^2 + 4 \).

3. The graph of \( f(x) = 0.5x^2 \) is shown.
   Sketch the graph of \( g(x) = 0.5(x + 1)^2 - 2 \).
Example 9

Let \( f(x) = -2(x + 6)^2 - 2 \).

1. Find the vertex on the graph of \( f \).

2. Find at least 4 other points on the graph of \( f \) – at least 2 points on each side of the vertex. Then sketch the graph of \( f \).

3. The domain of \( f \), in interval notation, is \( \) ________________

4. The range of \( f \), in interval notation, is \( \) ________________
Exercise #63
The graph \( f(x) = a(x - h)^2 + k \) is shown. Sketch each graph.

a. \( g(x) = a(x - 2h)^2 + k \)

b. \( p(x) = a(x - h)^2 + 2k \)

c. \( q(x) = 2a(x - h)^2 + k \)

d. \( r(x) = 2a(x - 2h)^2 + 2k \)
Example 10

Let \( f(t) \) represent the percentage of Americans who are pro-choice at \( t \) years since 1990.

a. Find the appropriate regression equation (linear exponential or quadratic) that models the data well.
   \[
y = f(t) = 0.56t^2 - 9.92t + 91.43
\]

b. Identify the vertex in the graph of \( f \) and explain its meaning in this application.

   The vertex is __________ which means the model predicts
   ________________________________________________
   ________________________________________________
   ________________________________________________

b. Identify the vertex in the graph of \( f \) and explain its meaning in this application.

   The y-intercept is __________ which means the model predicts
   ________________________________________________
   ________________________________________________
   ________________________________________________

d. Use the vertex and y-intercept to write the model in vertex form.

e. Predict the percentage of Americans who will be pro-choice in 2008.

f. Predict when 75% of Americans will be pro-choice.
7.2 Sketching Quadratic Functions in Standard Form

Symmetric Points
Two points on a parabola that have the same y-coordinate are called **symmetric points**. Since every pair of symmetric points is equidistant from the axis of symmetry of a parabola, **the x-coordinate of the vertex is the average of the x-coordinates of any pair of symmetric points** (see figure 1).

Example 1
The graph of $y = x^2 + 6x + 7$ is shown.
1. List three pairs of symmetric points.
2. Average the $x$-values of each pair of symmetric points. What is the equation of the axis of symmetry of the parabola?
3. Verify that the equation of the axis of symmetry can also be computed by $x = -\frac{b}{2a}$.
To sketch the graph of \( f(x) = ax^2 + bx + c \)

1. Find the \( y \)-intercept \((0, c)\).
2. Find the symmetric point to the \( y \)-intercept. Set \( f(x) = c \) and solve.
3. Find the vertex \( \left( \frac{-b}{2a}, f \left( \frac{-b}{2a} \right) \right) \)
   a. The \( x \)-coordinate of the vertex is the average of any two \( x \)-coordinates of symmetric points. Or the axis has the equation given by \( x = \frac{-b}{2a} \).
   b. The \( y \)-coordinate of the vertex is found by evaluating the function at the \( x \)-coordinate of the vertex.
4. Sketch the parabola.

Example 2
Let \( f(x) = x^2 - 4x + 7 \).

1. Find the \( y \)-intercept.
2. Find the symmetric point to the \( y \)-intercept.
3. Find the vertex.
4. Sketch the graph of \( f \).
Example 3
Let \( f(x) = -0.9x^2 - 5.8x - 5.7 \).
1. Find the \( y \)-intercept of \( f \) and its symmetric point.

2. Find the vertex \( \left( \frac{-h}{2a}, f\left( \frac{-h}{2a} \right) \right) \).

3. Sketch the graph of \( f \) by plotting at least 5 points.
Maximum or Minimum Values of a Quadratic Function

Let \( f(x) = ax^2 + bx + c \) have vertex \((h, k)\). Then

1. \( h = -\frac{b}{2a} \) and \( k = f(h) \).

2. If \( a < 0 \), then the parabola opens downward and the maximum value of \( f \) is \( k \) (see figure 1).

3. If \( a > 0 \), then the parabola opens upward and the minimum value of \( f \) is \( k \) (see figure 2).

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Example 5

The profit (in millions of dollars) for a company is given by

\[
P(t) = 0.09t^2 - 1.65t + 9.72 .
\]

where \( t \) is the number of years since 1990. Find the vertex of \( P \) and explain its meaning in this application.
7.3 Solving Quadratic Equations by Square Roots

The square root of a number \( a \), denoted \( \sqrt{a} = a^{\frac{1}{2}} \), is the number whose square is \( a \). In the expression \( \sqrt{a} \), \( a \) is the radicand and \( \sqrt{\phantom{a}} \) is the radical symbol. The expression \( \sqrt{a} \) is read “radical \( a \),” or “square-root of \( a \).” There are two square roots of any positive number - one positive and one negative. The positive square root is called the principal square root, denoted \( \sqrt{a} \). The negative square root is denoted \( -\sqrt{a} \).

Example 1
Evaluate each of the following.
1. \( \sqrt{36} \)
2. \( -\sqrt{81} \)
3. \( \sqrt{-25} \)

Product and Quotient Properties of Square Roots
If \( a \geq 0 \) and \( b \geq 0 \), then
1. **Product Property** \( \sqrt{ab} = \sqrt{a} \sqrt{b} \)
2. **Quotient Property** \( \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \)

**Caution:** There are no similar properties for addition or subtraction. i.e. \( \sqrt{a + b} \neq \sqrt{a} + \sqrt{b} \) and \( \sqrt{a - b} \neq \sqrt{a} - \sqrt{b} \)
A Radical Expression is Simplified if
1. The radicand contains no perfect square factors and no fractions.
2. There are no radicals in a denominator.
3. The numerator and denominator have no common factors (other than 1).

Example 2
Simplify each of the following.

1. \( \sqrt{18} \)
2. \( \sqrt{32} \)
3. \( \sqrt{\frac{x}{9}} \)

Example 3
Simplify each of the following.

1. \( \sqrt{7} \sqrt{7} \)
2. \( \sqrt{47} \sqrt{47} \)
3. \( \sqrt{x} \sqrt{x} \)
4. \( \sqrt{a + b} \sqrt{a + b} \)

Example 4
Simplify each of the following by rationalizing the denominator.

1. \( \frac{5}{\sqrt{3}} \)
2. \( \frac{-33}{\sqrt{11}} \)
Example 5
Simplify each of the following.
1. \( \sqrt{\frac{7}{8}} \) 
2. \( \sqrt{\frac{11}{20}} \)

Solving Equations in the form \( x^2 = a \)
Let \( a \) be a nonnegative number. Then
\[ x^2 = a \] is equivalent to \( x = \pm \sqrt{a} \)

Steps to Solve Equations by Extracting Square Roots
1. Isolate the Perfect Square Term.
2. Take the square root of both sides placing a \( \pm \) symbol on the numeric side.
3. Solve for \( x \).

Example 6
Solve each equation.
1. \( x^2 = 25 \)
2. \( x^2 - 50 = 0 \)
3. \( 3y^2 - 13 = 0 \)

Solve \( (x - 5)^2 - 2 = 14 \)
1. \( (x - 5)^2 = 16 \)
2. \( \sqrt{(x - 5)^2} = \pm \sqrt{16} \)
3. \( x - 5 = \pm 4 \)
   \( x = 5 \pm 4 \)
   \( x = 1, \ x = 9 \)
Example 7
Solve each equation for the exact [non-decimal] solution(s) w/o your calculators. Check your solutions.

1. \((2x + 7)^2 = 16\)
2. \(2(4x - 3)^2 + 5 = 11\)
3. \((6 - 5x)^2 = 0\)
4. \(5(6x - 11)^2 + 9 = 0\)

Example 8
Let \(f(x) = -3(x - 5)^2 + 14\).

1. Find the exact [non-decimal] value of the zeros of \(f\).
2. Write the \(x\)-intercepts to 2 decimal places.
Example 9  Using a Quadratic Model to Make Predictions

Let \( f(t) \) represent the average monthly participation (in millions) in the Food Stamp Program in the year that is \( t \) years since 1990 (see table).

1. Find the appropriate regression model (linear, quadratic or exponential) for the data. Round the constants to 2 decimal places.

<table>
<thead>
<tr>
<th>Year</th>
<th>Average Monthly Participation (millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1996</td>
<td>25.4</td>
</tr>
<tr>
<td>1998</td>
<td>19.8</td>
</tr>
<tr>
<td>2000</td>
<td>17.2</td>
</tr>
<tr>
<td>2002</td>
<td>19.1</td>
</tr>
<tr>
<td>2004</td>
<td>23.9</td>
</tr>
</tbody>
</table>

2. Estimate when the average monthly participation in the Food Stamp Program was 30 million.

3. Find the vertex of the model and explain its meaning in this application.
Complex Numbers
So far we have said the equation $x^2 = -1$ has no real solutions. However, by introducing complex numbers, we can solve such equations.

The Imaginary Unit $i$
The imaginary unit, written $i$, is the number whose square is $-1$. That is,

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}$$

Square Root of a Negative Number
If $p$ is a positive real number, then

$$\sqrt{-p} = i\sqrt{p}$$

For example, $\sqrt{-4} = \sqrt{-1 \cdot 4} = \sqrt{-1} \sqrt{4} = i \cdot 2 = 2i$. If $b$ is a non-zero real number, then $bi$ is an imaginary number.

Example 10
Write each number in $bi$ form.
1. $\sqrt{-18}$
2. $-\sqrt{-24}$
Complex and Imaginary Numbers
A complex number is a number in the form
\[ a + bi \]
where \( a \) and \( b \) are real numbers. If \( a = 0 \), then \( a + bi = bi \) is an imaginary number. If \( b = 0 \), then \( a + bi = a \) is a real number.

<table>
<thead>
<tr>
<th>Complex Number</th>
<th>( a )</th>
<th>( b )</th>
<th>Complex and Real or Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 + 5i )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( -\frac{3}{4} - \frac{1}{2}i )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( -i\sqrt{2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Solving Equations in the form \( x^2 = a \)
Let \( a \) be a real number. Then \( x^2 = a \) is equivalent to \( x = \pm \sqrt{a} \)

Example 11
Solve. Write complex solutions in \( a + bi \) form.
1. \( x^2 = -12 \)
2. \( (x + 3)^2 = -28 \)
3. \( 4w^2 + 25 = 17 \)
4. \( \left(x - \frac{1}{3}\right)^2 = -\frac{7}{4} \)
7.4 Solving Quadratic Equations by Completing the Square

A **perfect square trinomial** is the trinomial that results when a binomial is squared and written in expanded form. A **perfect square trinomial with a leading coefficient of 1** is obtained by squaring a binomial in the form \( x + k \) (note the coefficient on \( x \) is one). When the binomial \((x + k)\) is squared and expanded we get

\[
(x + k)^2 = x^2 + 2kx + k^2,
\]

and the following are true:
1. The leading coefficient of the resulting perfect-square trinomial is one.
2. The constant term of the perfect-square trinomial \((k^2)\) is the square of one-half of the linear coefficient \((2k)\).
3. The perfect square trinomial factors to \((x + k)^2\), where \(k\) is one-half the linear coefficient of the perfect square trinomial.

**Example 1**

For each show the constant term of the perfect-square trinomial is one-half of the coefficient of its linear term squared

1. \((x + 5)^2 = x^2 + 10x + 25\)
2. \((x - 7)^2 = x^2 - 14x + 49\)
3. \((x + 11)^2 = x^2 + 22x + 121\)
4. \((x - 3)^2 = x^2 - 6x + 9\)

**Example 2**

Find the value of \(c\) so the expression is a perfect square trinomial. Then factor each perfect square trinomial.

1. \(x^2 + 12x + c\)
2. \(x^2 - 8x + c\)
3. \(x^2 + 4x + c\)
Steps To Solve $ax^2 + bx + c = 0$ by Completing the Square

1. Write the equation in the form $ax^2 + bx = -c$. That is, get all the variable terms on one side of the equation and the constant term on the other side.

2. The completing the square procedure requires the leading coefficient to be one, so divide both sides by $a$. Then
   
   $$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. To make $x^2 + \frac{b}{a}x$ a perfect square trinomial, add half the linear coefficient squared to both sides of the equation.

4. Factor the perfect square trinomial:
   
   $$(x + \text{half the linear coefficient})^2$$

5. Solve by taking square roots.

6. Check your solutions.

   $$x = 3 \pm \sqrt{10}$$
   
   $$x = 3 \pm 10$$

   $$x = 3.1623, -6.1623$$

Solve

$$3x^2 - 18x - 3 = 0$$

1. $3x^2 - 18x = 3$

2. $\frac{3x^2 - 18x}{3} = \frac{3}{3}$

   $$x^2 - 6x = 1$$

Since $\left[\frac{1}{2}(-6)\right]^2 = (-3)^2 = 9$

3. $x^2 - 6x + 9 = 1 + 9$

   $$x^2 - 6x + 9 = 10$$

4. $(x - 3)^2 = 10$

5. $\sqrt{(x - 3)^2} = \pm \sqrt{10}$

   $$x - 3 = \pm \sqrt{10}$$

   $$x = 3 \pm \sqrt{10}$$

6. $x = 3.1623, -6.1623$

You can also verify the graph of $y = 3x^2 - 18x - 3$ has $x$-intercepts at $(-0.162, 0)$ and $(6.162, 0)$. 
Example 3

a. Solve \( x^2 - 8x + 9 = 0 \) by completing the square. Find the exact [non-decimal] solutions.

b. Write the solutions in decimal form and verify they correspond to the zeros of \( y = x^2 - 8x + 9 \)

Example 4

Solve \( 3x^2 + 7x - 5 = 0 \) by completing the square. Find the exact [non-decimal] solutions. Check your solutions.
Example 5
Solve by completing the square. Write complex solutions in $a + bi$ form. Check your solutions.

1. $m^2 + 5m + 7 = 0$

2. $3t^2 - 2t = -6$
7.5 Solving Quadratic Equations by the Quadratic Formula

Example 1
Solve by completing the square: \( ax^2 + bx + c = 0 \)

The Quadratic Formula
A quadratic equation in the form \( ax^2 + bx + c = 0 \) has solutions given by:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Example 2
Solve using the quadratic formula: \( 2x^2 - 7x - 15 = 0 \)
Example 3
Solve using the quadratic formula
1. \(5x(4x - 3) = -2\)

2. \(2x^2 + 12x = -18\)

3. \(x(x + 1) = -1\)
The Discriminant in the Quadratic Formula Determines
The Number of Real Solutions to \( ax^2 + bx + c = 0 \)

In the quadratic formula the radicand \( b^2 - 4ac \) is called the **discriminant**. The sign of the discriminant determines whether the quadratic equation \( ax^2 + bx + c = 0 \) has zero, one or two real solutions by:

(i) If \( b^2 - 4ac > 0 \), then there are **two real solutions** and two \( x \)-intercepts.

(ii) If \( b^2 - 4ac = 0 \), then there is **one real solution** and one \( x \)-intercept.

(iii) If \( b^2 - 4ac < 0 \), then there are **no real solutions** and no \( x \)-intercepts.

**Example 4**  **No Real Solutions, \( b^2 - 4ac < 0 \)**

1. Use the discriminant to determine the number of real solutions to \( 2x^2 - 3x + 5 = 0 \).
   \[
   b^2 - 4ac = (-3)^2 - 4(2)(5) = -31 < 0 \quad \text{No Real Solutions}
   \]

2. Find the zeros and \( x \)-intercepts of \( f(x) = 2x^2 - 3x + 5 \).

\[
Xscl = Yscl = 1
\]

**Example 5**  **Two Real Solutions, \( b^2 - 4ac > 0 \)**

1. Use the discriminant to determine the number of real solutions to \( 2x^2 - 4x - 3 = 0 \).

2. Find the zeros and \( x \)-intercepts of \( f(x) = 2x^2 - 4x - 3 \).

\[
Xscl = Yscl = 1
\]
Example 6  One Real Solutions, $b^2 - 4ac = 0$
1. Use the discriminant to determine the number of real solutions to $2x^2 - 4x + 2 = 0$.

2. Find the zeros and $x$-intercepts of $f(x) = 2x^2 - 4x + 2$.

Example 7
Let $f(x) = x^2 - 4x + 8$. Use the discriminate to find the number of points that lie on the graph of $f$ at each indicated height.
1. $y = 5$

2. $y = 4$

3. $y = 3$
Example 8
Let \( f(t) \) represent the number of U.S. hotel openings at \( t \) years since 1990.

1. Find the appropriate regression model (linear, exponential or quadratic) for \( f \) and predict the number of openings in 2005.

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Openings</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>1476</td>
</tr>
<tr>
<td>1998</td>
<td>1519</td>
</tr>
<tr>
<td>1999</td>
<td>1402</td>
</tr>
<tr>
<td>2000</td>
<td>1246</td>
</tr>
<tr>
<td>2001</td>
<td>1047</td>
</tr>
</tbody>
</table>

2. Find \( t \) when \( f(t) = 700 \)? Explain its meaning in this application.

3. Find the \( t \)-intercept of \( f \) and explain its meaning in this application.
7.6 Solving a System of Three Linear Equations to Find Quadratic Functions

Example 1

Without using your calculators, find the equation of the parabola that contains the points (0, 1), (3, 7), and (4, 5).

1. We need to find the value of \(a\), \(b\) and \(c\) in \(y = ax^2 + bx + c\). So substitute the three points above into the equation to get three equations with three unknowns \((a, b, c)\).

<table>
<thead>
<tr>
<th>Point</th>
<th>(y = ax^2 + bx + c)</th>
<th>System of Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 1))</td>
<td>(1 = a(0)^2 + b(0) + c)</td>
<td></td>
</tr>
<tr>
<td>((3, 7))</td>
<td>(7 = a(3)^2 + b(3) + c)</td>
<td></td>
</tr>
<tr>
<td>((4, 5))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Find the values of \(a\), \(b\) and \(c\) using elimination. Choose any two pairs of equations and eliminate one variable from them. Then you will have a system of two equations and two unknowns to solve.

3. Use the regression capabilities of your calculator to verify the result.
Example 2
Without using your calculators, find the equation of the parabola that contains the points (1, 1), (2, 3), and (3, 9).

<table>
<thead>
<tr>
<th>Point</th>
<th>$y = ax^2 + bx + c$</th>
<th>System of Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3, 9)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 3
Without using your calculators, find the equation of the parabola that contains the points (4, -3), (5, 2), and (6, 9).

<table>
<thead>
<tr>
<th>Point</th>
<th>$y = ax^2 + bx + c$</th>
<th>System of Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4, -3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 9)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 4
Solve the system of three equations in three variables. Express your solution as an ordered triple $(x, y, z)$.

\[
x + y - z = -1
\]
\[-4x - y + 2z = -7
\]
\[2x - 2y - 5z = 7
\]
7.7 Finding Regression Models

7.7 Homework: 1-11 (odd) When asked to find the equation that models the data, just find the appropriate (linear, exponential, or quadratic) regression model for the data.

Example 1
Although the number of property crimes increased from 1984 to 1991, it decreased after that. Let \( f(t) \) represent the number of property crimes (in millions) at \( t \) years since 1980. Draw a scattergram on your calculators and find the appropriate (linear, exponential, or quadratic) regression model for the data.

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Property Crimes (millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1984</td>
<td>10.61</td>
</tr>
<tr>
<td>1988</td>
<td>12.36</td>
</tr>
<tr>
<td>1992</td>
<td>12.51</td>
</tr>
<tr>
<td>1996</td>
<td>11.79</td>
</tr>
<tr>
<td>2000</td>
<td>9.90</td>
</tr>
</tbody>
</table>

Example 2
Sales of DVD players have increased over the years. Let \( f(t) \) represent the sales (in millions) of DVD players at \( t \) years since 1990. Draw a scattergram on your calculators and find the appropriate (linear, exponential, or quadratic) regression model for the data.

<table>
<thead>
<tr>
<th>Year</th>
<th>DVD Player Sales (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>0.3</td>
</tr>
<tr>
<td>1998</td>
<td>0.9</td>
</tr>
<tr>
<td>1999</td>
<td>3.6</td>
</tr>
<tr>
<td>2000</td>
<td>9.9</td>
</tr>
<tr>
<td>2001</td>
<td>16.0</td>
</tr>
</tbody>
</table>
7.8 Modeling with Quadratic Functions

Four Step Modeling Process
1. Create a scattergram of the data and decide whether a line, exponential curve, or parabola comes closest to the data.
   a. Enter the independent variable values into list 1 and the dependent variable values into list 2. To access the lists press STAT/EDIT.
   b. Turn STATPLOT 1 on and press ZOOM/9: ZoomStat to view the scattergram.
2. Find the appropriate regression model (linear, exponential or quadratic) of the function in the STAT/CALC menu.
3. Verify that the graph of the regression equation comes close to the data points. You may have to compute and graph more than one type of equation to determine the best model.
4. Use the model to draw conclusions, make estimates, and/or make predictions.

Example 1
Let \( f(t) = 1.04t^2 - 14.73t + 52.17 \) represent the sales (in millions) of DVD players during the year that is \( t \) years since 1990 (see table).

1. Find \( f(15) \) and explain its meaning in this application.

<table>
<thead>
<tr>
<th>Year</th>
<th>DVD Player Sales (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>0.3</td>
</tr>
<tr>
<td>1998</td>
<td>0.9</td>
</tr>
<tr>
<td>1999</td>
<td>3.6</td>
</tr>
<tr>
<td>2000</td>
<td>9.9</td>
</tr>
<tr>
<td>2001</td>
<td>16.0</td>
</tr>
</tbody>
</table>

2. Find \( t \) when \( f(t) = 100 \) and explain its meaning in this application.

3. In what years does there seem to be model breakdown?
Example 2
Let $p(t)$ represent the percentage of workers who use computers on the job at age $t$ years (see table).

1. Find the appropriate regression model (linear, exponential or quadratic) that matches the data well.

2. Estimate the age(s) at which half the workers use computers on the job.

2. Use $p$ to estimate the percentage of 22-year-old workers who use computers on the job.

3. Estimate the age of workers who are most likely (maximum percentage) to use computers on the job. What percentage of workers at this age use computers on the job?

4. Find the $t$-intercepts. What do they mean in this situation?
Example 3
The numbers of videocassettes and DVDs bought by U.S. dealers are given in the table for various years. Let \( v(t) \) and \( d(t) \) be the numbers (in millions of units) of videocassettes and DVDs, respectively since 1990.

1. Find the appropriate (linear, exponential, quadratic) regression equations for \( v(t) \) and \( d(t) \). Write the constants to two decimal places.

2. Estimate when sales of DVDs overtook sales of videocassettes.

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Videocassettes (millions of units)</th>
<th>Number of DVDs (millions of units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>57.0</td>
<td>1.6</td>
</tr>
<tr>
<td>1999</td>
<td>86.2</td>
<td>8.6</td>
</tr>
<tr>
<td>2000</td>
<td>99.4</td>
<td>13.9</td>
</tr>
<tr>
<td>2001</td>
<td>86.2</td>
<td>37.1</td>
</tr>
<tr>
<td>2002</td>
<td>73.6</td>
<td>79.3</td>
</tr>
<tr>
<td>2003</td>
<td>53.2</td>
<td>110.9</td>
</tr>
</tbody>
</table>
Example 4

Airlines originated frequent flier programs in 1981. The cumulative unredeemed miles are the total number of frequent-flier miles that members have not redeemed (see table). Let $c$ be the cumulative unredeemed miles (in trillions of miles) at $t$ years since 1980.

1. Find the appropriate regression equation (linear, quadratic, or exponential) that models the situation.

<table>
<thead>
<tr>
<th>Year</th>
<th>Cumulative Unredeemed Miles (trillions of miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1986</td>
<td>0.1</td>
</tr>
<tr>
<td>1988</td>
<td>0.4</td>
</tr>
<tr>
<td>1990</td>
<td>0.9</td>
</tr>
<tr>
<td>1992</td>
<td>1.5</td>
</tr>
<tr>
<td>1994</td>
<td>2.2</td>
</tr>
<tr>
<td>1996</td>
<td>3.2</td>
</tr>
<tr>
<td>1998</td>
<td>4.6</td>
</tr>
<tr>
<td>2000</td>
<td>6.6</td>
</tr>
<tr>
<td>2002</td>
<td>9.1</td>
</tr>
<tr>
<td>2004</td>
<td>12.4</td>
</tr>
<tr>
<td>2005</td>
<td>14.2</td>
</tr>
</tbody>
</table>

2. In what years is model breakdown certain?

3. Predict the cumulative unredeemed miles in 2010.

4. Predict when the cumulative unredeemed miles will be 25 trillion miles.
Example 5
A group charters a flight that normally costs $800 per person. A group discount reduces the fare by $10 for each ticket sold; the more tickets sold, the lower the per-person fare. There are 60 seats on the plane, including 4 for the crew. What size group would maximize the airline revenue?